A condition for separability of $\mathbf{C}(X{\to}\Omega)$ under the uniform topology

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Entrance. Spaces X and Ω are metric spaces and $\mathbf{C}(X{\to}\Omega)$ is the space of continuous functions $X{\to}\Omega$ equipped with the uniform norm $\|\cdot,\cdot\|$. A subset $E \subset X$ is δ***-dense*** if every point of X is within distance δ of E. Thus E is dense if it is δ-dense for all (positive numbers) δ. Recall that a metric space is ***separable*** if it has a *countable* dense subset. (My later notes call this property ***CSD***, countably self-dense.)

1: Theorem. *Suppose X and Ω are separable, with X compact. Then $\mathbf{C}(X{\to}\Omega)$ is separable.* ♢

Proof. Let $X' \subset X$ and $\Omega' \subset \Omega$ be countable dense subsets of their respective spaces. Say that a triple $(\varepsilon, \delta, \varphi)$ is ***acceptable*** if

i: Both ε and δ are reciprocals of positive integers.

ii: There is a *finite* set $F \subset X'$ which is δ-dense in X. Moreover, φ is a map from F into Ω'.

Say that an $f \in \mathbf{C}(X{\to}\Omega)$ is $(\varepsilon, \delta, \varphi)$***-good*** if

a: For all $x \in F$: $\quad |\varphi(x), f(x)| < \varepsilon$.

b: For all $x, y \in X$:

$$|x, y| < \delta \implies |f(x), f(y)| < \varepsilon.$$

Notice, for all $f \in \mathbf{C}(X{\to}\Omega)$ and all ε, that

2: *There exist δ and φ so that f is $(\varepsilon, \delta, \varphi)$-good.*

For since X is compact, f is uniformly continuous and so there is a sufficiently small $\delta = (1/n)$ which fulfills (b). Again by compactness there is a finite subset $F \subset X'$ which is δ-dense in X. Now define $\varphi\colon F{\to}\Omega'$ by, for each $x \in F$, simply picking a point in Ω', call it $\varphi(x)$, which is within distance ε of $f(x)$.

The distance estimate

Now suppose we have a second $(\varphi, \varepsilon, \delta)$-good function h. For each $y \in X$ there exists $x \in F := \mathrm{Dom}(\varphi)$ such that $|x, y| < \delta$. Thus the conclusion of (b) holds for both f and h, to yield that

$$*: \quad \big|h(y), f(y)\big| \leqslant |h(y), h(x)| + |h(x), \varphi(x)| + |\varphi(x), f(x)| + |f(x), f(y)| \leqslant 4\varepsilon\,.$$

Thus

$$\|h, f\| \leqslant 4\varepsilon\,.$$

The last step. There are but countably many acceptable triples $\tau := (\varepsilon, \delta, \varphi)$. For each such triple, pick a function h_τ which is τ-good. Then this collection $\{h_\tau\}_\tau$ is dense, courtesy of (2) and ($*$). ♦

Remark. The theorem certainly fails without the hypothesis of compactness: Let $X := \mathbb{N}$ and $\Omega := \{0, 1\}$, both equipped with with the discrete metric, $|n, m| := 1$ when $n \neq m$. Then the uniform norm is the discrete metric on the uncountable space $\mathbf{C}(X{\to}\Omega) \overset{\text{note}}{=\!=} \{0, 1\}^{\mathbb{N}}$. □

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